

# TO ONE PROBLEM OF SAUT-TEMAM FOR THE 3D ZAKHAROV-KUZNETSOV EQUATION

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ABSTRACT. An initial-boundary value problem for the 3D Zakharov-Kuznetsov equation posed on an unbounded domain is considered. Existence and uniqueness of a global regular solution as well as exponential decay of the  $H^2$ -norm for small initial data are proven.

## 1. INTRODUCTION

We are concerned with the existence, uniqueness and exponential decay of the  $H^2$ -norm for global regular solutions to an initial-boundary value problem (IBVP) for the 3D Zakharov-Kuznetsov (ZK) equation

$$u_t + (1 + u)u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0 \quad (1.1)$$

which describes the propagation of nonlinear ionic-sonic waves in a plasma submitted to a magnetic field directed along the  $x$  axis. This equation is a three-dimensional analog of the well-known Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0. \quad (1.2)$$

Equations (1.1), (1.2) are typical examples of so-called dispersive equations which attract considerable attention of both pure and applied mathematicians in the past decades. The KdV equation is probably most studied in this context. The theory of the initial-value problem (IVP henceforth) for (1.2) is considerably advanced today [1, 14, 15, 34, 37].

Recently, due to physics and numerics needs, publications on initial-boundary value problems to (1.2) both in bounded and unbounded domains for dispersive equations have appeared [2, 20, 25, 40]. In particular, it has been discovered that the KdV equation posed on a bounded interval possesses an implicit internal dissipation. This allowed to prove the exponential decay rate of small solutions for (1.2) posed on unbounded intervals without adding any artificial damping

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term [20]. Similar results were proved for a wide class of dispersive equations of any odd order with one space variable [12].

However, (1.2) is a satisfactory approximation for real waves phenomena while the equation is posed on the whole line ( $x \in \mathbb{R}$ ); if cutting-off domains are taken into account, (1.2) is no longer expected to mirror an accurate rendition of reality. The correct equation in this case (see, for instance, [2]) should be written as

$$u_t + u_x + uu_x + u_{xxx} = 0. \quad (1.3)$$

Indeed, if  $x \in \mathbb{R}$ ,  $t > 0$ , the linear traveling term  $u_x$  in (1.3) can be easily scaled out by a simple change of variables, but it can not be safely ignored for problems posed both on finite and semi-infinite intervals without changes in the original domain.

Once bounded domains are considered as a spatial region of waves propagation, their sizes appear to be restricted by certain critical conditions. We recall, however, that if the transport term  $u_x$  is neglected, then (1.3) becomes (1.2), and it is possible to prove the exponential decay rate of small solutions for (1.2) posed on any bounded interval. More results on control and stabilizability for the KdV equation can be found in [32, 33].

Later, the interest on dispersive equations became to be extended for multi-dimensional models such as Kadomtsev-Petviashvili (KP) and ZK equations. As far as the ZK equation is concerned, results both on IVP and IBVP can be found in [10, 11, 13, 27, 28, 29, 30, 35]. The biggest part of these publications is devoted to study of well-posedness of the Cauchy problem and initial-boundary value problems for the 2D ZK equation [10, 11, 13, 27, 28]. In the case of the 3D ZK equation, there are results on local well-posedness for the Cauchy problem [29, 30]; the existence of local strong solutions to an initial-boundary value problem posed on a bounded domain, [40], as well as the existence of global weak solutions [35].

Our work has been inspired by [35] where (1.1) posed on an unbounded domain was considered. A thorough analysis of these papers has revealed that an implicit dissipativity of the terms  $u_{xyy} + u_{xzz}$  may help to establish a global well-posedness of initial-boundary value problems in classes of regular solutions. Yearlier this dissipativity has been used in order to prove exponential decay for the 2D ZK equation [19, 26].

The main goal of our work is to prove the existence and uniqueness of global-in-time regular solutions of (1.1) posed on unbounded domains and the exponential decay rate of these solutions for sufficiently small

initial data. To cope with this problem, we exploited the strategy completely different from the standard schemes: first to prove the existence result and after that to study uniqueness and decay properties of solutions. In our case, we prove simultaneously existence of global regular solutions and their exponential decay.

The paper is outlined as follows. Section I is Introduction. Section 2 contains formulation of the problem and auxiliaries. In Section 3, we prove the existence of global regular solutions and, simultaneously, exponential decay of the  $H^2$ -norm. In section 4 uniqueness of a regular solutions and continuous dependence on initial data are proven.

## 2. PROBLEM, PRELIMINARES AND MAIN RESULT

Let  $L > 0$  be a finite number. Define  $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x \in (0, L), y \in \mathbb{R}; z \in \mathbb{R}\}$ ,  $\mathcal{S} = \mathbb{R}^2$ .

Consider in  $\Omega \times (0, t)$  the following initial- boundary value problem for the Zakharov-Kuznetsov equation:

$$Lu = u_t + u_x + uu_x + \Delta u_x = 0; \quad (2.1)$$

$$u(0, y, z, t) = u(L, y, z, t) = u_x(L, y, z, t) = 0, \quad (2.2)$$

$$u(x, y, z, 0) = u_0(x, y, z), \quad (x, y, z) \in \Omega. \quad (2.3)$$

Hereafter subscripts  $u_x$ ,  $u_{xy}$ , etc. denote the partial derivatives, as well as  $\partial_x$  or  $\partial_{xy}^2$  when it is convenient. Operators  $\nabla$  and  $\Delta$  are the gradient and Laplacian acting over  $\Omega$ . By  $(\cdot, \cdot)$  and  $\|\cdot\|$  we denote the inner product and the norm in  $L^2(\Omega)$ ,  $\|\cdot\|_{H^k}$  stands for the norm in  $L^2$ -based Sobolev spaces, and  $\|u\|_{[2]}^2 = \|u_{xx}\|^2 + \|u_{yy}\|^2 + \|u_{zz}\|^2$ .

**Theorem 2.1.** *Let  $L \leq \frac{\pi}{2}$  and  $u_0(x, y, z)$  satisfying the following conditions:*

$$u_0(0, x, y, z) = u_0(L, y, z, t) = u_{0x}(L, y, z, t) = 0,$$

$$\|u_0\|^2 + \|u_{0yy}\|^2 + \|u_{0yz}\|^2 + \|u_{0zz}\|^2 + J_0 < \infty,$$

$$\|u_0\|^4 \leq \frac{\pi^2}{8K_1L^2}, \quad J_0^2 \leq \frac{\pi^2}{200K_2L^2}, \quad (2.4)$$

where  $K_1 = 2^{16}3^3(1+L)(\frac{2^3}{25}C_1+1)$ ,  $C_1 = 2 + \frac{2^{13}}{3}\|u_0\|^4$ ,  $K_2 = 2^{19}3^3(1+L)^6$  and  $J_0 = ((1+x), |u_0 + u_{0x} + u_0u_{0x} + \Delta u_{0x}|^2)$ .

Then there exists a unique global regular solutions to (2.1 - 2.3):

$$u \in L^\infty(0, +\infty; H^2(\Omega)) \cap L^2(0, +\infty; H^3(\Omega)), \quad (2.5)$$

$$u_t \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)), \quad (2.6)$$

$$\Delta u_x \in L^\infty(0, +\infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)) \quad (2.7)$$

such that

$$\|u\|_{H^2(\Omega)}^2(t) + \|u_t\|^2(t) \leq C(L, J_0)e^{-\frac{\chi}{2}t}, \quad \forall t > 0, \quad (2.8)$$

where  $\chi = \frac{\pi^2}{2L^2(1+L)}$ .

We will need the following results:

**Lemma 2.1.** *Let  $u \in H^1(\Omega)$  and  $\gamma$  be the boundary of  $\Omega$ .*

*If  $u|_\gamma = 0$ , then*

$$\|u\|_{L^q(\Omega)} \leq 4^\theta \|\nabla u\|^\theta \|u\|^{1-\theta}, \quad (2.9)$$

where  $\theta = 3(\frac{1}{2} - \frac{1}{q})$ .

*If  $u|_\gamma \neq 0$ , then*

$$\|u\|_{L^q(\Omega)} \leq 4^\theta C_\Omega \|u\|_{H^1(\Omega)}^\theta \|u\|^{1-\theta}, \quad (2.10)$$

where  $C_\Omega$  does not depend on a size of  $\Omega$ .

*Proof.* See [16, 17].

**Lemma 2.2.** *Let  $v \in H_0^1(0, L)$ . Then*

$$\|v_x\|^2(t) \geq \frac{\pi^2}{L^2} \|v\|^2(t). \quad (2.11)$$

*Proof.* The proof is based on the Steklov inequality: let  $v(t) \in H_0^1(0, \pi)$ , then  $\int_0^\pi v_t^2(t) dt \geq \int_0^\pi v^2(t) dt$ . Inequality (2.11) follows by a simple scaling.  $\square$

**Lemma 2.3.** *Let  $f(t)$  be a continuous positive function such that*

$$f'(t) + (\alpha - kf^n(t))f(t) \leq 0. \quad (2.12)$$

$$\alpha - kf^n(0) > 0. \quad (2.13)$$

*Then*

$$f(t) < f(0) \quad (2.14)$$

*for all  $t > 0$ .*

*Proof.* Obviously,  $f'(0) + (\alpha - kf^n(0))f^n(0) \leq 0$ . Since  $f$  is continuous, there exists  $T > 0$  such that  $f(t) < f(0)$  for every  $t \in [0, T]$ . Suppose that there is  $\tau > 0$  and  $f(0) = f(\tau)$ . Integrating (2.12), we find

$$\int_0^\tau (\alpha - kf^n(t))f^n(t) dt < 0$$

that contradicts (2.13). Therefore,  $f(t) < f(0)$  for all  $t > 0$ .  
(See also [38]).

The proof of Lemma 2.2 is complete.  $\square$

### 3. EXISTENCE OF REGULAR SOLUTIONS

**Regularized problem.** To solve (2.1)-(2.3), we exploit the parabolic regularization of this problem as follows:

For  $\epsilon > 0$  (small), consider in  $\Omega \times (0, t)$  the following parabolic problem:

$$L_\epsilon u_m^\epsilon = u_{mt}^\epsilon + u_{mx}^\epsilon + u_m^\epsilon u_{mx}^\epsilon + \Delta u_{mx}^\epsilon + \epsilon \partial^4 u_m^\epsilon = 0, \quad (3.1)$$

$$\begin{aligned} u_m^\epsilon(0, y, z, t) &= u_m^\epsilon(L, y, z, t) = u_{mx}^\epsilon(L, y, z, t) \\ &= \epsilon u_{mxx}^\epsilon(0, y, z, t) = 0, \end{aligned} \quad (3.2)$$

$$u_m^\epsilon(x, y, z, 0) = u_{0m}(x, y, z), \quad (3.3)$$

$$u_{0m}(0, y, z) = \epsilon u_{0mxx}(0, y, z) = u_{0m}(L, y, z) = u_{0mx}(L, y, z) = 0, \quad (3.4)$$

where

$$\partial^4 u_m^\epsilon = \frac{\partial^4}{\partial x^4} u_m^\epsilon + \frac{\partial^4}{\partial y^4} u_m^\epsilon + \frac{\partial^4}{\partial z^4} u_m^\epsilon,$$

$u_0$  is an independent of  $\epsilon$  approximation of  $u_0$  such that for all  $m \in \mathbb{N}$ .

Define

$$J_{0m}^\epsilon = ((1+x), |u_{0m}^\epsilon + u_{0mx}^\epsilon + u_{0m}^\epsilon u_{0mx}^\epsilon + \Delta u_{0mx}^\epsilon + \epsilon \partial^4 u_{0m}^\epsilon|^2) \quad (3.5)$$

$$J_{0m} = ((1+x), |u_{0m} + u_{0mx} + u_{0m} u_{0mx} + \Delta u_{0mx}|^2). \quad (3.6)$$

It is known [17, 37, 36] that there exists a unique regular solution of (3.1)-(3.4), provided  $u_{0m}$  is sufficiently smooth.

Our goal is to obtain estimates for the  $u_m^\epsilon$  independent of  $m$  and  $\epsilon$  with  $u_{0m}$  sufficiently smooth, fixed; then to pass the limit as  $\epsilon$  tends to 0 getting a solution to (2.1)-(2.3) with initial data  $u_{0m}$ . After that we pass to the limit as  $m$  tends to  $\infty$  and  $u_{0m}$  tends to  $u_0$ , obtaining a solution to the original problem.

We will assume that  $u_{0m}$  converges to  $u_0$  in the following sense:

$$J_{0m} \rightarrow J_0, \quad \|u_{0myy}\| \rightarrow \|u_{0yy}\|, \quad \|u_{0myz}\| \rightarrow \|u_{0yz}\|, \quad \|u_{0mzz}\| \rightarrow \|u_{0zz}\|,$$

as  $m \rightarrow \infty$ .

We assume that  $m \geq m^*$ , where  $m^*$  is a natural number such that  $J_{0m} \leq 2J_0$ . In turn,  $\epsilon > 0$  is sufficiently small such that  $J_{0m}^\epsilon \leq 4J_0$ .

**Lemma 3.1.** *Under the conditions of Theorem 2.1, for  $m$  sufficiently large and  $\epsilon$  sufficiently small, the following independent of  $\epsilon$  and  $m$  estimates hold:*

$$\begin{cases} u_m^\epsilon \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\ u_{mx}^\epsilon \text{ is bounded in } L^2(0, T; L^2(\mathcal{S})). \end{cases} \quad (3.7)$$

*Proof. Estimate I.* Multiply (3.1) by  $u_m^\epsilon$  and integrate over  $\Omega \times (0, t)$  to obtain

$$\|u_m^\epsilon\|^2(t) + \int_0^t \int_{\mathcal{S}} (u_{mx}^\epsilon)^2(0, y, z, \tau) dy dz d\tau + 2\epsilon \int_0^t \|u_m^\epsilon\|_{[2]}^2(t) ds \leq \|u_{0m}\|^2$$

and for  $m$  sufficiently large

$$\|u_m^\epsilon\|^2(t) + \int_0^t \int_{\mathcal{S}} (u_{mx}^\epsilon)^2(0, y, z, \tau) d\tau ds \leq 2\|u_0\|^2. \quad (3.8)$$

**Estimate II.** Dropping the indices  $m, \epsilon$ , we transform the scalar product

$$2(L_\epsilon u_m^\epsilon, (1+x)u_m^\epsilon)(t) = 0 \quad (3.9)$$

into the following equality:

$$\begin{aligned} & \frac{d}{dt}((1+x), u^2)(t) + (1-\epsilon) \int_{\mathcal{S}} u_x^2(0, y, z, t) dy dz - \|u\|^2(t) \\ & + 3\|u_x\|^2(t) + \|u_y\|^2(t) + \|u_z\|^2(t) + 2\epsilon \mathcal{P}_1(t) = \frac{2}{3}(1, u^3)(t), \end{aligned} \quad (3.10)$$

where  $\mathcal{P}_1 = ((1+x), u_{xx}^2 + u_{yy}^2 + u_{zz}^2)(t)$ .

Making use of (2.9), we find

$$\begin{aligned} I = \frac{2}{3}(1, u^3)(t) & \leq \frac{2}{3}\|u\|_{L^3(\Omega)}^3(t) \leq \frac{2^4}{3}[\|\nabla u\|^{1/2}(t)\|u\|^{1/2}(t)]^3 \\ & \leq \frac{1}{2}\|\nabla u\|^2(t) + \frac{2^{11}}{3}\|u\|^6(t). \end{aligned} \quad (3.11)$$

By Lemma 2.2,

$$\|u_x\|^2(t) \geq \frac{\pi^2}{L^2}\|u\|^2(t). \quad (3.12)$$

Substituting (3.8), (3.11) and (3.12) into (3.10), we obtain for a fixed, sufficiently large  $m$  that

$$\frac{d}{dt}((1+x), u^2)(t) + \left[ \frac{2\pi^2}{L^2} - 1 - \frac{2^{13}}{3}\|u_0\|^4 \right] \|u\|^2(t) \leq 0.$$

Under conditions of Theorem 2.1, we have

$$\frac{\pi^2}{L^2} - 1 - \frac{2^{13}}{3} \|u_0\|^4 \geq 0. \quad (3.13)$$

Hence

$$\frac{d}{dt}((1+x), u^2)(t) + \frac{\pi^2}{L^2}((1+x), u^2)(t) \leq 0$$

and

$$\|u_m^\epsilon\|^2(t) \leq 2(1+L)\|u_0\|^2 e^{-\chi t}, \quad (3.14)$$

where  $\chi = \frac{\pi^2}{L^2(1+L)}$ . Returning to (3.10), using (3.11) and (3.13), we obtain

$$\int_0^t \left\{ \|\nabla u_m^\epsilon\|^2(\tau) + \int_S u_{mx}^\epsilon(0, y, z, \tau) dy dz \right\} d\tau \leq 2(1+L)\|u_0\|^2.$$

**Lemma 3.2.** *Under the conditions of Theorem (2.1), for  $m$  sufficiently large, the following independent of  $\epsilon$  and  $m$  estimates hold:*

$$\begin{cases} u_m^\epsilon \text{ is bounded in } L^\infty(0, T; H^1(\Omega)), \\ u_{mt}^\epsilon \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \end{cases}$$

*Proof.* **Estimate III.** Dropping the indices  $\epsilon$ ,  $m$ , transform the inner product

$$2(L_\epsilon u_m^\epsilon, (1+x)u_m^\epsilon)(t) = 0 \quad (3.15)$$

into the equality

$$\begin{aligned} & \int_S u_x^2(0, y, z, t) dy dz + 3\|u_x\|^2(t) + \|u_y\|^2(t) + \|u_z\|^2(t) \\ & + 2\epsilon \mathcal{P}_1 = \frac{2}{3}(1, u^3)(t) + \|u\|^2(t) - 2((1+x)u, u_t)(t). \end{aligned} \quad (3.16)$$

By Holder and Young inequalities,

$$\begin{aligned} 2 \int_\Omega (1+x)uu_t d\Omega & \leq 2\|u\|(t) \left( \int_\Omega [(1+x)u_t]^2 d\Omega \right)^{1/2} \\ & \leq \|u\|^2(t) + \int_\Omega (1+x)^2 u_t^2 d\Omega \\ & \leq \|u\|^2(t) + (1+L)((1+x), u_t^2)(t). \end{aligned}$$

Making use of (2.9), (3.14) and the last inequality, we reduce (3.16) to the form

$$\|\nabla u\|^2(t) \leq 2\left[2 + \frac{2^{11}}{3}\|u\|^4(t)\right]\|u\|^2(t) + 2((1+L)(1+x), (u_t)^2)(t)$$

$$\leq 4(1+L)C_1 e^{-xt} + 2(1+L)((1+x), |u_{mt}^\epsilon|^2)(t), \quad (3.17)$$

where  $C_1 = 2 + \frac{2^{13}}{3}\|u_0\|^4$  is independent of  $t > 0$ .

Returning to (3.16), we get

$$\begin{aligned} \|u_{mx}^\epsilon\|^2(t) &\leq \frac{2}{5}(2 + \frac{2^{13}}{3}\|u_0\|^4)\|u_m^\epsilon\|^2(t) \\ &+ \frac{2}{5}(1+L)((1+x), |u_{mt}^\epsilon|^2)(t). \end{aligned} \quad (3.18)$$

**Estimate IV.** Consider the inner product

$$2((L_\epsilon u_m^\epsilon)_t, (1+x)u_{mt}^\epsilon)(t) = 0. \quad (3.19)$$

We calculate

$$2((1+x), u_t(uu_x)_t) = ((1+x)u_x + u, u_t^2) \quad (3.20)$$

$$\leq \|(1+x)u_x + u\| \|u_t\|_{L^4(\Omega)}^2. \quad (3.21)$$

Exploiting Lemma 2.1 and the Young inequality, we obtain

$$\begin{aligned} 2((1+x), (uu_x)_t)(t) &\leq \frac{1}{8}\|\nabla u_t\|^2(t) + (1+L)^4 3^3 2^{16} [\|u_x\|^4(t) \\ &+ \|u\|^4(t)] \|u_t\|^2(t). \end{aligned} \quad (3.22)$$

Substituting (3.20) and (3.22) into (3.19), we get

$$\begin{aligned} \frac{d}{dt}((1+x), (u_t)^2)(t) &+ [\frac{23}{8}\|u_{xt}\|^2(t) + \frac{7}{8}\|u_{zt}\|^2(t) + \frac{7}{8}\|u_{yt}\|^2(t)] \\ &- (1 + (1+L)^4 3^3 2^{16} [\|u_x\|^4(t) + \|u\|^4(t)] \|u_t\|^2(t)) \leq 0. \end{aligned} \quad (3.23)$$

Since by (3.18),

$$\|u_x\|^4(t) \leq \frac{2^2}{25}C_1^2\|u\|^4(t) + \frac{2^2}{25}(1+L)^2((1+x), u_t^2)^2(t)$$

and by Lemma 2.2,

$$\|u_{xt}\|^2(t) \geq \frac{\pi^2}{L^2}\|u_t\|^2(t),$$

then (3.23) reads

$$\begin{aligned} \frac{d}{dt}((1+x), |u_t|^2)(t) &+ [\frac{\pi^2}{L^2} - 1 - 2^{16}3^3(1+L)^4(\frac{2^3}{25}C_1^2 + 1)\|u_0\|^4 \\ &- 2^{19}3^3(1+L)^6(1+x, |u_t|^2)^2(t)] \|u_t\|^2(t) \leq 0. \end{aligned}$$



According to Theorem 2.1 notations,

$$\begin{aligned} & \frac{d}{dt}((1+x), |u_{mt}^\epsilon|^2)(t) + \left[ \frac{\pi^2}{L^2} - 1 - K_1 \|u_0\|^4 \right. \\ & \quad \left. - K_2((1+x), |u_{mt}^\epsilon|^2)^2(t) \right] \|u_{mt}^\epsilon\|^2(t) \leq 0. \end{aligned} \quad (3.24)$$

For  $\epsilon$  small and  $m$  sufficiently large fixed  $2\epsilon^2((1+x), |\partial^4 u_{0m}^\epsilon|^2) \leq J_0$  and

$$|J_{0m}^\epsilon|^2 \leq 5J_0^2 \quad (3.25)$$

By Lemma 2.3,  $((1+x), |u_{mt}^\epsilon|^2)(t) \leq J_{0m}^\epsilon$  for all  $t > 0$  and making use of (3.24), (3.25), we obtain

$$\frac{d}{dt}((1+x), |u_{mt}^\epsilon|^2)(t) + \left[ \frac{\pi^2}{L^2} - 1 - K_1 \|u_0\|^4 - 25K_2 J_0^2 \right] \|u_t^\epsilon\|^2(t).$$

By (2.4) and (3.25), we get  $\frac{\pi^2}{2L^2} - 1 - K_1 \|u_0\|^4 - 25K_2 J_0^2 \geq 0$ .

Hence

$$\frac{d}{dt}((1+x), |u_{mt}^\epsilon|^2)(t) + \frac{\pi^2}{2L^2}((1+x), |u_{mt}^\epsilon|^2)(t) \leq 0$$

and

$$\|u_{mt}^\epsilon\|^2(t) \leq 2(1+L)J_0 e^{-\frac{\pi}{2}t}. \quad (3.26)$$

Returning to (3.23), we obtain

$$\int_0^t \|\nabla u_{mt}^\epsilon\|^2(\tau) d\tau < C(J_0, L) \quad (3.27)$$

and from (3.14), (3.17),

$$\|u_m^\epsilon\|_{H^1(\Omega)}^2(t) \leq C(L, J_0) e^{-\frac{\pi}{2}t}, \quad \forall t > 0. \quad (3.28)$$

for  $\epsilon > 0$  sufficiently small and  $m$  sufficiently large fixed.

**Lemma 3.3.** *Under the assumptions of Theorem 2.1, we find*

$$\begin{cases} u_{mxy}^\epsilon, u_{mzx}^\epsilon, u_{myy}^\epsilon, u_{mzz}^\epsilon, u_{myz}^\epsilon \text{ are bounded in } L^\infty(0, T; L^2(\Omega)), \\ u_m^\epsilon \text{ is bounded in } L^2(0, T; H^2(\Omega)), \\ \nabla u_{myy}^\epsilon, \nabla u_{mzz}^\epsilon, \nabla u_{mxyz}^\epsilon \text{ are bounded in } L^2(0, T; L^2(\Omega)), \\ u_{mx}^\epsilon(0, y, z, t) \text{ is bounded in } L^\infty(0, T; H^1(\mathcal{S})) \cap L^2(0, T; H^2(\mathcal{S})). \end{cases}$$

*Proof. Estimate V.* Dropping the indices  $\epsilon$ ,  $m$ , transform the scalar product

$$-2((1+x)Au_m^\epsilon, u_{myy}^\epsilon + u_{mzz}^\epsilon)(t) = 0$$

into the following equality:

$$\begin{aligned} & \|u_y\|^2(t) + \|u_z\|^2(t) + (1-2\epsilon) \int_S [u_{xy}^2(0, y, z, t) + u_{xz}^2(0, y, z, t)] dy dz \\ & + \|u_{yy}\|^2(t) + \|u_{zz}\|^2(t) + 2\|u_{yz}\|^2(t) + 3\|u_{xy}\|^2(t) + 3\|u_{xz}\|^2(t) \\ & + ((1+x)u_x - u, u_y^2)(t) + ((1+x)u_x - u, u_z^2)(t) \\ & + 2\epsilon\mathcal{P}_2(t) = 2((1+x)u_t, u_{yy} + u_{zz})(t), \end{aligned} \quad (3.29)$$

where  $\mathcal{P}_2(t) = ((1+x), |u_{xxy}^2 + u_{xxz}^2 + u_{yyy} + u_{yyz}^2 + u_{zzz}^2 + u_{zzy}^2|)(t)$ .

We estimate

$$\begin{aligned} I_1 &= ((1+x)u_x - u, u_y^2)(t) \leq \|(1+x)u_x - u\|(t) \|u_y\|_{L^4(\Omega)}^2(t) \\ &\leq (1+L) [\|u_x\|(t) + \|u\|(t)] 4^{3/2} C_\Omega^2 \|\nabla u\|^{1/2}(t) \|\nabla u_y\|^{3/2}(t) \\ &\leq \frac{1}{8} \|\nabla u_y\|^2(t) + (1+L)^4 C_\Omega^8 2^{16} 3^3 [\|u_x\|^4(t) + \|u\|^4(t)] \|\nabla u\|^2(t). \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= ((1+x)u_x - u, u_z^2)(t) \leq \frac{1}{8} \|\nabla u_z\|^2(t) \\ &+ (1+L)^4 C_\Omega^8 2^{16} 3^3 [\|u_x\|^4(t) + \|u\|^4(t)] \|\nabla u\|^2(t). \end{aligned}$$

Substituting  $I_1, I_2$  into (3.29), we find

$$\begin{aligned} & \int_S [u_{xy}^2(0, y, z, t) + u_{xz}^2(0, y, z, t)] dy dz + \|u_{yy}\|^2(t) + \|u_{zz}\|^2(t) \\ & + \|u_{yz}\|^2(t) + \|u_{xy}\|^2(t) + \|u_{xz}\|^2(t) \\ & \leq C_4(L) [\|\nabla u\|^6(t) + \|u\|^4(t) \|\nabla u\|^2(t) + ((1+x), u_t^2)(t)]. \end{aligned}$$

Making use of (3.28),

$$\begin{aligned} & \|u_{mxy}^\epsilon\|^2(t) + \|u_{mzx}^\epsilon\|^2(t) + \|u_{myy}^\epsilon\|^2(t) + \|u_{myz}^\epsilon\|^2(t) + \|u_{mzz}^\epsilon\|^2(t) \\ & + \int_S \{(u_{mxy}^\epsilon)^2(0, y, z, t) + (u_{mzx}^\epsilon)^2(0, y, z, t)\} dy dz \\ & \leq C(J_0, L) e^{-\frac{\lambda}{2}t}. \end{aligned} \quad (3.30)$$

To prove that  $u$  is bounded in  $L^\infty(0, T; H^2(\Omega))$ , it is sufficiently to estimate  $\|u_{mxx}^\epsilon\|(t)$ .

**Estimate VI.** From the inner product

$$2(L_\epsilon u_m^\epsilon, (1+x)[u_{myyyy}^\epsilon + u_{myyzz}^\epsilon + u_{mzzzz}^\epsilon])(t) = 0, \quad (3.31)$$

dropping the indices  $\epsilon$  and  $m$ , we find

$$\begin{aligned}
J_1 &= 2 \int_{\Omega} u_t(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega \\
&= \frac{d}{dt} ((1+x), u_{yy}^2 + u_{yz}^2 + u_{zz}^2)(t), \\
J_2 &= 2 \int_{\Omega} u_x(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega \\
&= \int_{\Omega} (u_{yy}^2 + u_{yz}^2 + u_{zz}^2) d\Omega, \\
J_3 &= 2 \int_{\Omega} u_{xxx}(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega = 3 [\|u_{yyx}\|^2(t) \\
&\quad + \|u_{xyz}\|^2(t) + \|u_{zzx}\|^2(t)] + \int_{\mathcal{S}} \{u_{yyx}^2(0, y, z, t) + u_{xyz}^2(0, y, z, t) \\
&\quad + u_{zzx}^2(0, y, z, t)\} dy dz, \\
J_4 &= 2 \int_{\Omega} u_{yyx}(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega \\
&= \|u_{yyy}\|^2(t) + \|u_{zzy}\|^2(t) + \|u_{yyz}\|^2(t), \\
J_5 &= 2 \int_{\Omega} u_{zzx}(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega \\
&= \|u_{zzz}\|^2(t) + \|u_{yyz}\|^2(t) + \|u_{zzz}\|^2(t), \\
J_6 &= 2\epsilon \int_{\Omega} \partial_y^4 u(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega \\
&= \epsilon((1+x), u_{yyy}^2 + u_{yyzz}^2 + u_{yyyz}^2)(t), \\
J_7 &= 2\epsilon \int_{\Omega} \partial_z^4 u(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega \\
&= \epsilon((1+x), u_{zzzz}^2 + u_{yyzz}^2 + u_{zzzy}^2)(t), \\
J_8 &= 2\epsilon \int_{\Omega} \partial_x^4 u(1+x) [\partial_y^4 u + \partial_y^2 \partial_z^2 u + \partial_z^4 u] d\Omega = \epsilon((1+x), u_{yyxx}^2 \\
&\quad + u_{xxyz}^2 + u_{zzxx}^2)(t) - 2\epsilon \int_{\mathcal{S}} \{u_{yyx}^2(0, y, z, t) + u_{xyz}^2(0, y, z, t) \\
&\quad + u_{zzx}^2(0, y, z, t)\} dy dz.
\end{aligned}$$

Substituting  $J_1$ - $J_8$  into (3.31), we get

$$\frac{d}{dt} ((1+x), u_{yy}^2 + u_{zz}^2 + u_{yz}^2)(t) - [\|u_{yy}\|^2(t) + \|u_{zz}\|^2(t) + \|u_{yz}\|^2(t)]$$

$$\begin{aligned}
& + (1 - 2\epsilon) \int_S \{u_{xyy}^2(0, y, z, t) + u_{xzz}^2(0, y, z, t) + u_{xyz}^2(0, y, z, t)\} dydz \\
& + \|u_{yyy}\|^2(t) + \|u_{zzz}\|^2(t) + 2\|u_{yzz}\|^2(t) + 2\|u_{zyy}\|^2(t) + 3\|u_{xyy}\|^2(t) \\
& + 3\|u_{xzz}\|^2(t) + 3\|u_{xyz}\|^2(t) + \epsilon \mathcal{P}_3(t) + 2((1+x)uu_x, \partial_y^4 u)(t) \\
& + 2((1+x)uu_x, \partial_z^4 u)(t) + 2((1+x)uu_x, \partial_y^2 \partial_z^2 u)(t) = 0, \tag{3.32}
\end{aligned}$$

where  $\mathcal{P}_3 = ((1+x), [u_{yyy}^2 + 2u_{yyzz}^2 + u_{yyyz}^2 + u_{zzzz}^2 + u_{zzzy}^2 + u_{yyxx}^2 + u_{zzxx}^2])(t)$ .

We estimate the nonlinear term in the following manner:

$$\begin{aligned}
& 2 \int_{\Omega} uu_x(1+x)u_{yyy} d\Omega = -2 \int_{\Omega} u_y u_x(1+x)u_{yy} d\Omega \\
& - 2 \int_{\Omega} uu_{xy}(1+x)u_{yy} d\Omega \\
& = 2 \int_{\Omega} u_{yy}^2 u_x(1+x) d\Omega + 4 \int_{\Omega} u_y u_{xy}(1+x)u_{yy} d\Omega \\
& + 2 \int_{\Omega} uu_{yyx}(1+x)u_{yy} d\Omega \\
& = 2 \int_{\Omega} u_{yy}^2 u_x(1+x) d\Omega + 2 \int_{\Omega} (u_y^2)_x(1+x)u_{yy} d\Omega \\
& + \int_{\Omega} u_{yy}^2 u_x(1+x) d\Omega \\
& = \int_{\Omega} u_{yy}^2 u_x(1+x) d\Omega - 2 \int_{\Omega} u_y^2 u_{yy} d\Omega - 2 \int_{\Omega} u_y^2(1+x)u_{yyx} d\Omega \\
& - 2 \frac{1}{2} \int_{\Omega} uu_{yy}^2 d\Omega \\
& = ((1+x)u_x - u, u_{yy}^2) - 2(u_y^2, u_{yy}) - 2((1+x)u_y^2, u_{yyx}) \\
& = I_1 - I_2 - I_3. \tag{3.33}
\end{aligned}$$

Making use of (3.33), we find

$$\begin{aligned}
I_1 & = ((1+x)u_x - u, u_{yy}^2)(t) \\
& \leq \|(1+x)u_x - u\|(t) \|u_{yy}\|_{L^4(\Omega)}^2(t) \\
& \leq (1+L)[\|u_x\|(t) + \|u\|(t)] 4^{3/2} \|u_{yy}\|^{1/2}(t) \|\nabla u_{yy}\|^{3/2}(t) \\
& \leq \frac{3}{4} \delta_1^{4/3} \|\nabla u_{yy}\|^2(t) + \frac{2^{16}}{\delta_1^4} \|u_{yy}\|^2(t) [\|u_x\|(t) + \|u\|]^4(t) \\
& = \frac{3}{4} \delta_1^{4/3} \|\nabla u_{yy}\|^2(t) + \frac{2^{16}}{\delta_1^4} \|u_{yy}\|^2(t) [\|u_x\|^4(t) + \|u\|^4(t)]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{3}{4}\delta_1^{4/3}\|\nabla u_{yy}\|^2(t) + Ce^{-\frac{\chi}{2}t}, \\
I_2 &= -2(u_y^2, u_{yy}) \\
&\leq 2\|u_y\|^4(t) + 2\|u_{yy}\|^2(t) \leq 2\|u_y\|_{L^4(\Omega)}^4(t) + 2\|u_{yy}\|^2(t) \\
&\leq 4^4\|u_y\|\|\nabla u\|^3(t) + 2\|u_{yy}\|^2(t) \\
&\leq 2^8\left(\frac{1}{2}\|u_y\|^2(t) + \frac{1}{2}\|\nabla u\|^6(t)\right) + 2\|u_{yy}\|^2(t) \\
&\leq Ce^{-\frac{\chi}{2}t}, \\
I_3 &= -2((1+x)u_y^2, u_{yyx})(t) \\
&\leq 2\|u_y\|_{L^4(\Omega)}^2(t)\|u_{yyx}\|(t) \\
&\leq (1+L)4^{3/2}\|u_y\|^{1/2}(t)\|\nabla u_y\|^{3/2}(t)\|u_{yyx}\|(t) \\
&\leq \delta_2^2\|u_{yyx}\|^2(t) + \frac{2^8}{\delta_2^2}\|u_y\|^2(t)\|\nabla u_y\|^3(t) \\
&\leq \delta_2^2\|u_{yyx}\|^2(t) + \frac{2^7}{\delta_2^2}(\|u_y\|^2(t) + \|\nabla u_y\|^6(t)) \\
&\leq \delta_2^2\|u_{yyx}\|^2(t) + Ce^{-\frac{\chi}{2}t},
\end{aligned}$$

where  $\delta_1, \delta_2$  are arbitrary positive constants and  $C$  is a constant independent of  $\epsilon, m$  and  $t$ . Substituting  $I_1$ - $I_3$  into (3.33), we get

$$\begin{aligned}
&\int uu_x(1+x)u_{yyy} d\Omega \leq \frac{3}{4}\delta_1^{4/3}\|u_{yy}\|^2(t) + \delta_2^2\|u_{yyx}\|^2(t) \\
&+ Ce^{-\frac{\chi}{2}t},
\end{aligned} \tag{3.34}$$

Similarly,

$$\begin{aligned}
&\int uu_x(1+x)u_{zzzz} d\Omega \leq \frac{3}{4}\delta_1^{4/3}\|u_{zz}\|^2(t) + \delta_2^2\|u_{zzx}\|^2(t) \\
&+ Ce^{-\frac{\chi}{2}t}
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
2 \int_{\Omega} uu_x(1+x)u_{yyzz}(t) d\Omega &= -2((1+x)u_z u_x, \partial_y^2 \partial_z u)(t) \\
&- 2((1+x)u u_{zx}, \partial_y^2 \partial_z u)(t) = 2((1+x)u_x, u_{zy}^2)(t) \\
&- ((1+x)u_z^2, u_{xyy})(t) - (u_z^2, u_{yy})(t) + 2((1+x)u u_{xzz}, u_{yy})(t)
\end{aligned} \tag{3.36}$$

$$\equiv J_1 + J_2 + J_3 + J_4. \tag{3.37}$$

Using (3.17), (3.30), we estimate

$$J_1 = 2((1+x)u_x, u_{yz}^2)(t) \leq 2(1+L)\|u_x\|(t)\|u_{yz}\|_{L^4(\Omega)}^2(t)$$

$$\begin{aligned}
&\leq 2^4(1+L)\|u_x\|(t)\|u_{yz}\|^{1/2}(t)\|\nabla u_{yz}\|^{3/2}(t) \\
&\leq \frac{3}{4}\delta_3^{4/3}\|\nabla u_{yz}\|^2(t) + \frac{2^{14}}{\delta_3^4}(1+L)^4\|u_x\|^4(t)\|u_{yz}\|^2(t) \\
&\leq \frac{3}{4}\delta_3^{4/3}\|\nabla u_{yz}\|^2(t) + Ce^{-\frac{\chi}{2}t}, \\
J_2 &= -((1+x)u_z^2, u_{xyy})(t) \leq (1+L)\|u_{xyy}\|(t)\|u_z\|_{L^4(\Omega)}^2(t) \\
&\leq \delta_4\|u_{xyy}\|^2(t) + \frac{4^3(1+L)^2}{\delta_4}\|\nabla u\|(t)\|\nabla u_z\|^3(t) \\
&\leq \delta_4\|u_{xyy}\|^2(t) + Ce^{-\frac{\chi}{2}t}, \\
J_3 &= -(u_z^2, u_{yy})(t) \leq \|u_{yy}\|(t)\|u_z\|_{L^4(\Omega)}^2(t) \\
&\leq \|u_{yy}\|^2(t) + 4^2\|u_z\|(t)\|\nabla u_z\|^3(t) \\
&\leq \|u_{yy}\|(t)\|u_z\|_{L^4(\Omega)}^2(t) \leq \|u_{yy}\|^2(t) + Ce^{-\frac{\chi}{2}t}, \\
J_4 &= 2((1+x)uu_{xzz}, u_{yy})(t) \\
&\leq 2(1+L)\|u_{xzz}\|(t)\|u\|_{L^4(\Omega)}(t)\|u_{yy}\|_{L^4(\Omega)}(t) \leq \delta_5\|u_{xzz}\|^2(t) \\
&\quad + \frac{4^3(1+L)^2}{\delta_5}\|\nabla u_{yy}\|^{3/2}(t)\|u\|^{1/2}(t)\|\nabla u\|^{3/2}(t)\|u_{yy}\|^{1/2}(t) \\
&\leq \delta_5\|u_{xzz}\|^2(t) \\
&\quad + \frac{3\delta_6^{4/3}}{4\delta_5}\|\nabla u_{yy}\|^2(t) + \frac{4^{11}(1+L)^8}{\delta_5\delta_6^4}\|u\|^2(t)\|u_{yy}\|^2(t)\|\nabla u\|^6(t) \\
&\leq \delta_5\|u_{xzz}\|^2(t) + \frac{3\delta_6^{4/3}}{4\delta_5}\|\nabla u_{yy}\|^2(t) + Ce^{-\frac{\chi}{2}t},
\end{aligned}$$

where  $\delta_3, \delta_4$  are arbitrary positive constants and  $C$  is a constant independent of  $\epsilon, m$  and  $t$ . Substituting  $J_1$ - $J_4$  into (3.37) and making use of (3.32), (3.33), we reduce (3.32) to the form

$$\begin{aligned}
&\frac{d}{dt}((1+x), u_{yy}^2 + u_{zz}^2 + u_{yz}^2)(t) \\
&\quad + \int_{\mathcal{S}} \{u_{xyy}^2(0, y, z, t) + u_{xzz}^2(0, y, z, t) + u_{xyz}^2(0, y, z, t)\} dydz \\
&\quad + \|u_{yyy}\|^2(t) + \|u_{zzz}\|^2(t) + \|u_{yzz}\|^2(t) + \|u_{zyy}\|^2(t) \\
&\quad + \|u_{xyy}\|^2(t) + \|u_{xzz}\|^2(t) + \|u_{xyz}\|^2(t) \\
&\quad + \epsilon((1+x), [u_{yyy}^2 + 2u_{yyzz}^2 + u_{yyyz}^2 + u_{zzzz}^2 \\
&\quad + u_{zzzy}^2 + u_{yyxx}^2 + u_{zzxx}^2])(t) \leq C(J_0, L)e^{-\frac{\chi}{2}t}.
\end{aligned}$$

Integrating over  $(0, t)$ , we obtain

$$\begin{aligned}
& ((1+x), (u_{myy}^\epsilon)^2 + (u_{mzz}^\epsilon)^2 + (u_{myz}^\epsilon)^2)(t) + \int_0^t \left\{ \int_{\mathcal{S}} [(u_{mxyy}^\epsilon)^2(0, y, z, \tau) \right. \\
& + (u_{mxzz}^\epsilon)^2(0, y, z, \tau) + (u_{mxyz}^\epsilon)^2(0, y, z, \tau)] dydz \\
& + \|u_{myyy}^\epsilon\|^2(\tau) + \|u_{mzzz}^\epsilon\|^2(\tau) + \|u_{myzz}^\epsilon\|^2(\tau) + \|u_{mzyy}^\epsilon\|^2(\tau) \\
& + \|u_{mxyy}^\epsilon\|^2(\tau) + \|u_{mxzz}^\epsilon\|^2(\tau) + \|u_{mxyz}^\epsilon\|^2(\tau) \\
& + \epsilon((1+x), [(u_{myyyy}^\epsilon)^2 + 2(u_{myyzz}^\epsilon)^2 + (u_{myyyz}^\epsilon)^2 + (u_{mzzzz}^\epsilon)^2 \\
& + (u_{mzzzy}^\epsilon)^2 + (u_{myyxx}^\epsilon)^2 + (u_{mzzxx}^\epsilon)^2]) \left. \right\}(\tau) d\tau \\
& \leq C(L, J_0)((1+x), u_{0yy}^2 + u_{0zz}^2 + u_{0zy}^2)
\end{aligned} \tag{3.38}$$

for  $\epsilon$  sufficiently small and  $m$  fixed and sufficiently large, with the constant  $C(L, J_0)$  independent of  $t > 0$ .

**Estimate VII.** Dropping the indices  $\epsilon, m$  and the variables  $y, z, t$ , rewrite (3.1)-(3.4) in the form

$$u_{xxx} + uu_x + \epsilon u_{xxx} = g, \tag{3.39}$$

$$u(0) = u_{xx}(0) = u(1) = u_x(1) = 0. \tag{3.40}$$

By (3.38),  $g \in L^2(0, T; L^2(\Omega))$ . Multiplyng (3.39) by  $x$  and integrating over  $x \in (0, 1)$ , we get

$$u_x(0) + u_{xx}(1) - \frac{1}{2} \int_0^1 u^2 dx - \epsilon u_{xx}(1) + \epsilon u_{xxx}(1) = \int_0^1 xg dx. \tag{3.41}$$

Integrating (3.39) over  $(x, 1)$  gives

$$u_{xx}(1) - u_{xx}(x) - \frac{1}{2}u^2(x) + \epsilon u_{xxx}(1) - \epsilon u_{xxx}(x) = \int_x^1 g dx. \tag{3.42}$$

Subtracting (3.42) from (3.41), we obtain

$$u_x(0) - \frac{1}{2} \int_0^1 u^2 dx - \epsilon u_{xx}(1) + u_{xx} + \frac{1}{2}u^2 + \epsilon u_{xxx} = \int_0^1 gx dx - \int_x^1 g d\xi.$$

Define

$$h(x) = -u_x(0) + \frac{1}{2} \int_0^1 u^2 dx - \frac{1}{2}u^2 + \int_0^1 gx dx - \int_x^1 g d\xi.$$

Then (3.39) reads

$$u_{xx} + \epsilon u_{xxx} = \epsilon u_{xxx}(1) + h. \tag{3.43}$$

Multiplying (3.43) by  $u_{xx}$  and integrating over  $(0, 1)$ , we find

$$\begin{aligned}\epsilon \int_0^1 u_{xxx} u_{xx} dx &= \frac{\epsilon}{2} u_{xx}^2(1), \\ \epsilon \int_0^1 u_{xx}(1) u_{xx} dx &= \epsilon u_{xx}(1) \int_0^1 u_{xx} dx = -\epsilon u_{xx}(1) u_x(0).\end{aligned}$$

Hence, (3.43) becomes

$$\begin{aligned}\int_0^1 u_{xx}^2 dx + \frac{\epsilon}{2} u_{xx}^2(1) &= -\epsilon u_{xx}(1) u_x(0) + \int_0^1 u_{xx} h dx \\ &\leq \frac{\epsilon}{4} u_{xx}^2(1) + 4\epsilon u_x^2(0) + \frac{1}{2} u_{xx}^2 + \frac{1}{2} \int_0^1 h^2 dx.\end{aligned}$$

Therefore

$$\int_0^1 u_{xx}^2 dx + \frac{\epsilon}{4} u_{xx}^2(1) \leq 4\epsilon u_x^2(0) + \frac{1}{2} \int_0^1 h^2 dx \quad (3.44)$$

and

$$\|u_{mxx}^\epsilon\|^2(t) \leq C(J_0, L, \|u_{0yy}\|, \|u_{0yz}\|, \|u_{0zz}\|).$$

**Passage to the limit as  $\epsilon \rightarrow 0$ .** Using the estimates obtained in Lemmas 3.1, 3.2 3.3 and compactness arguments, we can pass to the limit as  $\epsilon \rightarrow 0$  in (3.1-3.4) and get a solution  $u_m$  for (2.1)-(2.3) with initial data  $u_{0m}$  for  $m$  large and fixed:

$$u_{mt} + (1 + u_m)u_{mx} + u_m u_{mx} + \Delta u_{mx} = 0; \quad (3.45)$$

$$u_m(0, y, z, t) = u_m(L, y, z, t) = u_{mx}(L, y, z, t) = 0, \quad (3.46)$$

$$u_m(x, y, z, 0) = u_{0m}(x, y, z), \quad (x, y, z) \in \Omega \quad (3.47)$$

such that

$$u_m \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad (3.48)$$

$$u_{mt} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \text{ for all } t > 0. \quad (3.49)$$

Rewriting (3.45) as

$$u_{mxxx} = -u_{mt} - u_{myyx} - u_{mzzx} - u_{mx} - u_m u_{mx},$$

and making use of (3.26), (3.27), (3.38), we find that

$$u_{mxxx} \in L^2(0, T; L^2(\Omega)), \text{ for all } T > 0.$$

**Passage to the limit as  $m \rightarrow \infty$ .** Since the constants of estimates in Lemmas 3.1, 3.2 3.3 do not depend on  $\epsilon, m, t$ , we can pass to the



limit in (3.45)-(3.47) as  $m \rightarrow \infty$  and obtain a solution  $u(x, y, z, t)$  for (2.1)-(2.3) such that

$$u \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)), \quad (3.50)$$

$$u(0, y, z, t) \in L^\infty(0, T; H^1(\mathcal{S})) \cap L^2(0, T; H^2(\mathcal{S})), \quad (3.51)$$

$$\begin{aligned} & \|u\|_{H^1(\Omega)}^2(t) + \|u_y\|_{H^1(\Omega)}^2(t) + \|u_z\|_{H^1(\Omega)}^2(t) + \int_{\mathcal{S}} \{u_{xy}^2(0, y, z, t) \\ & + u_{xz}^2(0, y, z, t)\} dy dz \leq C(J_0, L)e^{-\frac{\chi}{2}t}, \text{ for all } t > 0. \end{aligned} \quad (3.52)$$

**Regularity of u.** Making use of (3.32) and (3.52), write (2.1)-(2.3) in the form

$$\Delta u_x = -u_t - u_x - \frac{1}{2}(u^2)_x, \quad (3.53)$$

$$u_x(L, y, z, t) = 0, \quad (3.54)$$

$$u_x(0, y, z, t) = \phi(y, z, t) \in L^\infty(0, T; H^1(\mathcal{S})) \cap L^2(0, T; H^2(\mathcal{S})). \quad (3.55)$$

Denoting  $v = u_x - \phi(y, z, t)(L - x)$ , we get

$$\Delta v = -u_t - v - \frac{1}{2}(u^2)_x + \Delta(\phi(y, z, t)(L - x)) \equiv F(x, y, z, t), \quad (3.56)$$

where  $F \in L^\infty(0, T, H^{-1}(\Omega))$ .

From the inner product

$$(\Delta v, v) = (F, v),$$

we calculate

$$\begin{aligned} & \int_{\Omega} (\phi_y(x - L))_y (u_x - \phi(x - L)) d\Omega \leq \frac{3}{2} \int_{\mathcal{S}} \phi_y^2 dy dz + \frac{1}{2} \|u_{xy}\|^2(t), \\ & \int_{\Omega} (\phi_z(x - L))_z (u_x - \phi(x - L)) d\Omega \leq \frac{3}{2} \int_{\mathcal{S}} \phi_z^2 dy dz + \frac{1}{2} \|u_{xz}\|^2(t) \end{aligned}$$

and come to the inequality

$$\begin{aligned} & \|v_x\|^2(t) + \|v_y\|^2(t) + \|v_z\|^2(t) \leq \|u_t\|^2(t) + \frac{1}{2} \|u_x\|^2(t) \\ & + \frac{1}{2} \int_{\mathcal{S}} u_x^2(0, y, z, t) dy dz + \frac{3}{2} \|u_x\|^2(t) + \frac{1}{2} \int_{\mathcal{S}} u_x^2(0, y, z, t) dy dz \\ & + \frac{1}{4} \|u\|_{L^4(\Omega)}^4(t) + \frac{1}{4} \|v_x\|^2(t) + \frac{1}{2} \|u_x\|^2(t) + \int_{\mathcal{S}} u_x^2(0, y, z, t) dy dz \\ & + \frac{3}{2} \int_{\mathcal{S}} u_{xy}^2(y, z, t) + u_{xz}^2(y, z, t) dy + \frac{1}{2} \|u_{xy}\|^2(t) + \frac{1}{2} \|u_{xz}\|^2(t). \end{aligned}$$

Since  $\|u\|_{L^4}^4(t) \leq 4^3 \|u\|(t) \|\nabla u\|^3(t)$ , making use of (3.52), we get

$$\|u_{xx}\|^2(t) \leq C(J_0, L)e^{-\frac{\lambda}{2}t} \text{ for all } t > 0.$$

Returning to (3.53)-(3.55), we can see that  $F \in L^2(0, \infty; L^2(\Omega))$ . This implies that  $u \in L^2(0, \infty; H^2(\Omega))$ . Hence  $u \in L^\infty(0, \infty; H^2(\Omega)) \cap L^2(0, \infty; H^3(\Omega))$ .

**Regularity of  $\Delta u_x$ .** Writing  $\Delta u_x = -u_t - u_x - uu_x$ , and recalling that

$$u_x, u_t, uu_x \in L^\infty(0, T; L^2(\Omega)),$$

we estimate

$$\begin{aligned} \|uu_x\|_{H^1(\Omega)}(t) &\leq \|uu_x\|(t) + \|\nabla(uu_x)\|(t) \\ &\leq \|u\|_{L^4(\Omega)}(t) \|u_x\|_{L^4(\Omega)}(t) + \|u\|_{L^4(\Omega)}(t) \|\nabla u_x\|_{L^4(\Omega)}(t) + \|\nabla u\|_{L^4(\Omega)}^2(t) \\ &\leq 2^3 [\|u\|^{1/4}(t) \|\Delta u\|(t) \|\nabla u_x\|^{3/4}(t) \\ &\quad + \|u\|^{1/4}(t) \|\nabla u\|^{3/4}(t) \|\nabla u_x\|^{1/4}(t) \|\Delta u_x\|^{3/4}(t) \|\Delta u\|^{3/2}(t)] \\ &\quad + \|\nabla u\|^{1/4}(t) \|\Delta u\|^{3/4}(t) \leq C \|u\|_{H^3(\Omega)}^2(t). \end{aligned} \quad (3.57)$$

where  $C$  is a constant independent of  $t$ . By (3.48), (3.49) and (3.57) read

$$u_t, u_x, uu_x \in L^2(0, T; H^1(\Omega)).$$

Hence

$$\Delta u_x \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1(\Omega)).$$

This proves the existence part of Theorem 2.1.

#### 4. UNIQUENESS OF A REGULAR SOLUTION AND CONTINUOUS DEPENDENCE ON INITIAL DATA

**Theorem 4.1.** *A global regular solution to (2.1)-(2.3) is uniquely defined.*

*Proof.* Let  $u_1, u_2$  be two distinct solutions to (2.1)-(2.3) and  $w = u_1 - u_2$ . Then

$$\begin{aligned} ww_x + (wu_2)_x &= (u_1 - u_2)(u_1 - u_2)_x + (u_1 - u_2)_x u_2 + (u_1 - u_2) u_{2x} \\ &= u_1 u_{1x} - u_1 u_{2x} - u_2 u_{1x} + u_2 u_{2x} + u_{1x} u_2 - u_{2x} u_2 + u_1 u_{2x} - u_2 u_{2x} \\ &= u_1 u_{1x} - u_2 u_{2x} \end{aligned}$$

and (2.1)-(2.3) can be rewritten in the form

$$Lw = w_t + w_x + \Delta w_x + ww_x + (wu_2)_x, \quad (4.1)$$

$$w(0, y, z, t) = w(L, y, z, t) = w(L, y, z, t) = 0, \quad (4.2)$$

$$w(x, y, z, 0) = 0. \quad (4.3)$$

Transform the inner product

$$2((1+x)Lw, w)(t) = 0$$

into the following equality

$$\begin{aligned} & \frac{d}{dt}((1+x), w^2)(t) + \int_{\mathcal{S}} w_x^2(0, y, z, t) dy dz + 2\|w_x\|^2(t) + \|\Delta w\|^2(t) \\ & - \frac{2}{3}(1, w^3)(t) + ((1+x)u_{2x} - u_2, w^2)(t) = 0. \end{aligned}$$

Using Lemma 2.1, we find

$$I_1 = -\frac{2}{3}(1, w^3)(t) \leq \frac{3}{4}\delta^{4/3}\|\nabla w\|^2(t) + \frac{2^{14}}{3^4\delta^4}\|w\|^6(t)$$

and

$$\begin{aligned} I_2 &= ((1+x)u_{2x} - u_2, w^2)(t) \leq \|(1+x)u_{2x} - u_2\|(t)\|w\|_{L^4}^2(t) \\ &\leq (1+L)[\|u_{2x}\|(t) + \|u_2\|(t)]4^{3/2}\|w\|^{1/2}(t)\|\nabla w\|^{3/2}(t) \\ &\leq \frac{C}{\delta^4}(\|u_{2x}\|^4(t) + \|u_2\|^4(t)\|w\|^2(t)) + \frac{3}{4}\delta^{4/3}\|\nabla w\|^2(t). \end{aligned}$$

For  $\delta > 0$  sufficiently small, we find

$$\frac{d}{dt}(1+x, w^2)(t) \leq C[\|u_{2x}\|^4(t) + \|u_1\|^4(t) + \|u_2\|^4(t)](1+x, w^2)(t).$$

By the Grownwall Lemma,

$$\|w\|^2(t) \leq ((1+x), w^2)(t) \equiv 0.$$

**Remark 4.1.** If  $w(x, y, z, 0) = w_0(x, y, z) \neq 0$ , then

$$\|w\|^2(t) \leq ((1+x), w^2)(t) \leq C(L, J_0)((1+x), w_0^2) \quad \forall t > 0.$$

*This means continuous dependence of solutions to (2.1)-(2.3) on initial data.*

**Remark 4.2.** *The geometrical restriction  $L \leq \frac{\pi}{2}$  in Theorem 2.1 is caused by the presence of the term  $u_x$  in (2.1) and is connected with spectral properties of the linear spatial operator  $u_x + \Delta u_x$  and existing of critical size domains (see [19] in 2D case). On the other hand, there are some boundary conditions under which there are not critical size domains [8]. We need also small initial data in order to suppress destabilizing effects of the nonlinear convective term  $uu_x$ . We must note that in [21] such restrictions for  $c_s u_x$  and initial data did not appear while establishing the existence of weak solutions for the 3D ZK equation, but there was an open problem, still unresolved, on uniqueness of this weak solution.*

**Conclusions.** We have established the existence and uniqueness of global regular solutions to (2.1)-(2.3) as well as exponential decay of the  $H^2$ -norm exploiting an approach of proving simultaneously existence and exponential decay. Therefore geometrical restrictions and “smallness” conditions for initial data have appeared. Of course, these restrictions are not necessary while proving only existence and uniqueness of global regular solutions for the 2D ZK equation. Nevertheless, similar restrictions appear while proving exponential decay of the existing global regular solutions [19].

## REFERENCES

- [1] Bona J. L., Smith R. W.: The initial-value problem for the Korteweg-de Vries equation, *Phil. Trans. Royal Soc. London Series A*, 278, 555–601 (1975)
- [2] Bona, J. L., Sun, S. M., Zhang, B.-Y.: A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain, *Comm. Partial Differential Equations*, 28, 1391–1436 (2003)
- [3] Bourgain, E. A.: On the compactness of the support of solutions of dispersive equations, *Int. Math. Res. Notices* 9, 437–447 (1997)
- [4] Bubnov, B. A.: Solvability in the large of nonlinear boundary-value problems for the Kortewegde Vries equation in a bounded domain (Russian), *Differentsial'nye uravneniya* 16 (1980), 34–41. Engl. transl. in: *Diff. Equations* 16, 24–30 (1980)
- [5] Colin, T. and Gislson, M.: An initial-boundary-value problem that approximate the quarter-plane problem for the Korteweg-de Vries Equation, *Nonlinear Analysis* 46, 869–892 (2001)
- [6] Doronin, G. G., Larkin, N. A.: KdV equation in domains with moving boundaries, *J. Math. Anal. Appl.*, 328, 503–515 (2007)
- [7] Doronin, G. G., Larkin, N. A.: Stabilization of regular solutions for the Zakharov-Kuznetsov equation posed on bounded rectangles and on a strip, *Proceeding of the Edinburgh Mathematical Society*, 2(58), N3, 661–682 (2015)

- [8] Doronin, G. G., Larkin, N. A.: Stabilization for the linear Zakharov-Kuznetsov equation without critical size restrictions, *JMAA* 428, 337-355 (2015) <http://dx.doi.org/10.1016/j.jmaa.2015.03.010>
- [9] Doronin, G. G., Larkin, N. A.: Exponential decay for the linear Zakharov-Kuznetsov equation without critical domain restrictions, *Applied Mathematical Letters*, 27 6–10 (2014)
- [10] Faminskii, A. V.: The Cauchy problem for the Zakharov-Kuznetsov equation (Russian), *Differentsialnye Uravneniya*, 31, 1070–1081 (1995), Engl. transl. in: *Differential Equations*, 31, 1002–1012 (1995)
- [11] Faminskii, A. V.: Well-posed initial-boundary value problems for the Zakharov-Kuznetsov equation, *Electronic Journal of Differential equations*, 127, 1–23 (2008)
- [12] Faminskii, A. V., Larkin, N. A.: Initial-boundary value problems for quasilinear dispersive equations posed on a bounded interval, *Elec. J. Diff. Equations*, 2010, 1–20 (2010)
- [13] Farah, L. G., Linares, F., Pastor, A.: A note on the 2D generalized Zakharov-Kuznetsov equation: Local, global, and scattering results, *J. Differential Equations*, 253, 2558-2571 (2012)
- [14] Kato, T.: On the Cauchy problem for the (generalized) Korteweg-de Vries equations, *Advances in Mathematics Supplementary Studies*, *Stud. Appl. Math.*, 8, 93–128 (1983)
- [15] Kenig, C. E., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation and the contraction principle, *Commun. Pure Appl. Math.*, 46 527–620 (1993)
- [16] Ladyzhenskaya, O. A.: "The Boundary Value Problems of Mathematical Physics," *Applied Math. Sci.* 49, Springer-Verlag, New York, 1985.
- [17] Ladyzhenskaya, O. A. Solonnikov, V. A., Ural'tseva, N. N. : "Linear and Quasilinear Equations of Parabolic Type," American Mathematical Society, Providence, Rhode Island, 1968.
- [18] Lannes, D., Linares, F. and Saut, J.-C.: The Cauchy problem for the Euler-Poisson system and justification of the Zakharov-Kuznetsov equation, in *Studies in Phase Space Analysis with Applications to PDEs*, Series Progress in Nonlinear Differential Equations and Applications vol. 84, M. Cicognani, F. Colombini, D. Del Santo EDS., Birkhauser, 183–215 (2013), 795–806 (2002)
- [19] Larkin, N. A.: Exponential decay of the  $H^1$ -norm for the 2D Zakharov-Kuznetsov equation on a half-strip, *J. Math. Anal. Appl.*, 405, 326–335 (2013)
- [20] Larkin, N. A.: Korteweg-de Vries and Kuramoto-Sivashinsky Equations in Bounded Domains, *J. Math. Anal. Appl.*, 297, 169–185 (2004)
- [21] Larkin, N. A.: Global regular solutions for the 3d Zakharov-Kuznetsov equation posed on unbounded domains, *Journal of Mathematical Physics* 56, 091508 (2015) DOI:10.1063/1.4928924
- [22] Larkin, N. A.: The 2D Kawahara equation on a half-strip, *Applied Mathematics and optimization*, 70, 443–468 (2014) DOI: 101007 / 500245-0149246-4
- [23] Larkin, N.A. A: 2d Zakharov-Kuznetsov-Burguers equation with variable dissipation on a stripe, *EJDE*, NEO 1–20, 2015 (2015)
- [24] Larkin N. A., Luchesi, J.: General mixed problems for the KdV equations on bounded intervals, *EJDE*, vol. 2010, No 168, 1–17 (2010)

- [25] Larkin, N. A., Tronco, E.: Nonlinear quarter-plane problem for the Korteweg-de Vries equation, *Electron. J. Differential Equations*, 2011, 1–22 (2011)
- [26] Larkin, N. A. and Tronco, E.: Regular solutions of the 2D Zakharov-Kuznetsov equation on a half-strip, *J. Differential Equations*, 254, 81–101 (2013)
- [27] Linares, F. and Pastor, A.: Local and global well-posedness for the 2D generalized Zakharov-Kuznetsov equation, *J. Funct. Anal.*, 260, 1060–1085 (2011)
- [28] Linares, F. , Pastor, A., Saut, J.-C.: Well-posedness for the ZK equation in a cylinder and on the background of a KdV Soliton, *Comm. Part. Diff. Equations*, 35, 1674–1689 (2010)
- [29] Linares, F., Saut, J.-C.: The Cauchy problem for the 3D Zakharov-Kuznetsov equation, *Disc. Cont. Dynamical Systems, A* 24, 547–565 (2009)
- [30] Ribaud, F., Vento, S.: Well-posedness results for the three-dimensional Zakharov-Kuznetsov equation, *SIAM J. Math. Anal.*, 44, 2289–2304 (2012)
- [31] Rivas, I., Usman, M. and Zhang, B.-Y.: Global well-posedness and asymptotic behavior of a class of initial-boundary value problem for the Korteweg-de Vries equation on a finite domain, *Math. Control Related Fields*, 1 61–81 (2011)
- [32] Rosier, L.: A survey of controllability and stabilization results for partial differential equations, *RS - JESA*, 41, 365–411 (2007)
- [33] Rosier, L., Zhang, B.-Y.: Control and stabilization of the KdV equation: recent progress, *J. Syst. Sci. Complexity*, 22, 647–682 (2009)
- [34] Saut, J. C.: Sur quelques généralisations de l'équation de Korteweg-de Vries (French), *J. Math. Pures Appl.*, 58, 21–61 (1979)
- [35] Saut, J.-C., Temam, R.: An initial boundary-value problem for the Zakharov-Kuznetsov equation, *Advances in Differential Equations*, 15 (2010), 1001–1031.
- [36] Saut, J.-C., Temam, R., Wang, C.: An initial and boundary-value problem for the Zakharov-Kuznetsov equation in a bounded domain, *J. Math. Phys.*, 53, 115612 (2012) doi:10.1063/1.4752102
- [37] Temam, R.: Sur un problème non linéaire (French), *J. Math. Pures Appl.*, 48, 159–172 (1969)
- [38] Temam, R.: *Navier-Stokes Equations*, Ams Chelsea Publishing, Providence, Rhode Island, 2001 (2001)
- [39] Zakharov, V. E. and Kuznetsov, E. A.: On three-dimensional solitons, *Sov. Phys. JETP*, 39, 285–286 (1974)
- [40] Wang, C.: Local existence of strong solutions to the 3D Zakharov-Kuznetsov equation in a bounded domains, *Appl. Math. Optim.*, 69, 1–19 (2014)
- [41] Zhang, B.-Y.: Exact boundary controllability of the Korteweg-de Vries equation, *SIAM J. Control Optim.* 37, 543–565 (1999)

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